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Periodic Solutions of a Nonlinear Non-Dissipative Wave Equation

P. H. Rabinowitz

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PERIODIC SOLUTIONS OF A NONLINEAR NON-DISSIPATIVE WAVE EQUATION

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ABSTRACT

This paper concerns an existence proof of solutions of the partial differential equation

$$u_{tt} - u_{xx} + \varepsilon F(x,t,u) = 0 ,$$

where F is 2π periodic in t, under the boundary and periodicity conditions $u(0,t) = u(\pi,t) = 0$, $u(x,t+2\pi) = u(x,t)$. The main assumption is $\partial F/\partial u \geq \beta > 0$ and ε is sufficiently small. The main difficulty is solving an associated infinite dimensional bifurcation equation.

It is also shown how to find a solution by an expansion method if F is real analytic in u and a convergence proof for the formal series constructed is supplied.

§1. Introduction.

We consider the partial differential equation

$$u_{tt} - u_{xx} + \varepsilon F(x,t,u) = 0$$

where F is 2π periodic in t. We seek a solution satisfying the periodicity and boundary conditions $u(x,t+2\pi) = u(x,t)$ and $u(0,t) = u(\pi,t) = 0$. Our main result is a proof of the existence of such a solution provided F depends monotonically on u (or more precisely $\partial F/\partial u \geq \beta > 0$) and ε is sufficiently small.

For $\varepsilon = 0$, our equation reduces to $u_{tt} - u_{xx} \equiv \Box 0 = 0$. All solutions of this equation which satisfy the boundary conditions are 2π periodic. However, for $\varepsilon \neq 0$ we expect only one solution to the partial differential equation. The main difficulty in the problem is to find one such smooth solution of $\Box u = 0$ from which the solutions to the perturbed problem bifurcate. It turns out we can describe this particular solution of $\Box u = 0$ by a variational problem, namely $\underset{\Box u = 0}{\text{minimize}} \int_0^{2\pi}\int_0^{\pi} H(x,t,u)\,dx\,dt$ where $H_u = F$ and we only admit solutions of $\Box u = 0$ which satisfy the boundary conditions.

In §2 we formulate and solve the bifurcation problem by a variational argument. It is here that the monotonicity of F is crucial. In §3 we solve a related linear problem. The nonlinear equation is solved in §4 by an iteration scheme.

If F is assumed to be analytic in u, an expansion method can be devised in a natural manner to solve the partial differential equation. This is done in §5. The convergence of the formal series obtained is proved in §6 by a majorant method. In the Appendix some inequalities which are used are proved.

The question of finding periodic solutions for nonlinear partial differential equations arises in a natural manner as an extension of the analogous problem for ordinary differential equations. Stoker [1] suggested a nonlinear wave equation problem as a generalization of the ordinary differential equations theory of Poincare for finding a periodic solution of a system of ordinary differential equations in the neighborhood of a known periodic solution. Ficken and Fleishman [2] studied a dissipative wave equation by converting it to a nonlinear integral equation. Vejvoda [3] investigated a non-dissipative wave equation with the aid of special methods to solve the associated bifurcation equation.[1] Prodi [4], Cesari [5], and others have also studied such equations.

This paper represents part of the author's doctoral dissertation. The results presented here are an interesting offshoot of the main line of research which concerned periodic solutions of nonlinear dissipative hyperbolic partial differential equations. A forthcoming paper will present these results in addition to extensions of the results of this paper. The latter include investigations of nonlinearities which depend on derivatives, free vibrations, and stability.

[1]Stewart [9] announced a proof of the existence of a solution to a non dissipative wave equation, but as he informed the author, it was never published.

The author would like to express his gratitude to Professor Jurgen Moser for his encouragement and advice.

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We conclude the introduction with some notation. Let C^∞ denote the space of continuous infinitely differentiable real valued functions in x and t, and 2π periodic in t, $0 \leq x \leq \pi$. Let C_0^∞ be the subspace of C^∞ consisting of functions with support contained in $(0,\pi)$ with respect to x. Let H_0 be the completion of C^∞ with respect to $|\phi|_0^2 \equiv |\phi|^2 = \int_0^{2\pi}\int_0^{\pi} |\phi(x,t)|^2\, dx\, dt$. Let H_j be the completion of C^∞ with respect to $|\phi|_j^2 = \int_0^{2\pi}\int_0^{\pi} \sum_{|\sigma|=0}^{j} |D^\sigma\phi|^2\, dx\, dt$ where $\sigma = (\sigma_1,\sigma_2)$, $|\sigma| = \sigma_1 + \sigma_2$, $D^\sigma = (\partial^{\sigma_1+\sigma_2})/(\partial x^{\sigma_1}\partial t^{\sigma_2})$. Let $\hat{H}_j$ be the completion of C_0^∞ with respect to $|\ |_j$. All of the above subscripted H spaces are Hilbert spaces with the natural inner product. We will denote the H_0 inner product by $(\ ,\ )$ and the H_k inner product by $(\ ,\ )_k$. Let C_j be the space of j times continuously differentiable functions in x and t, and 2π periodic in t, $0 \leq x \leq \pi$. We use $\|\phi\|_j$ to denote the norm in C_j. $\|\phi\|_j = \sum_{|\sigma|=0}^{j} \|D^\sigma\phi\|$ where $\|\phi\| = \sup_{x,t} |\phi(x,t)|$.

§2. Formulation and solution of the bifurcation equation.

Let $\phi(x,t) \in C_2$ and $\phi_{tt} - \phi_{xx} = 0$. Then $\phi(x,t) = p(x+t) + q(-x+t)$. If ϕ also satisfies our boundary conditions, $\phi(x,t) = p(x+t) - p(-x+t)$ where p is 2π periodic for $\phi(0,t) = 0$ implies $p(t) = -q(t)$ and $\phi(\pi,t) = 0$ implies $p(\pi+t) = p(-\pi+t)$. Thus ϕ automatically satisfies the periodicity condition.

Let N denote the closure in H_0 of the set of such ϕ. $N = \{\phi(x,t) \in H_0 \mid \phi(x,t) = p(x+t) - p(-x+t)$, p 2π periodic in t, and $\int_0^{2\pi} p^2(a)\, da < +\infty\}$.

Using Fourier series, we can also characterize N as $N = \{\phi(x,t) \in H_0 \mid \phi(x,t) = \sum_{-\infty}^{\infty} a_j \sin jx\, e^{ijt}$ with $\sum_{-\infty}^{\infty} |a_j|^2 < +\infty\}$.

We try for a solution of our nonlinear partial differential equation of the form $u(x,t;\varepsilon) = v(x,t;\varepsilon) + \varepsilon w(x,t;\varepsilon)$ where $v \in N$ and $w \in N^{\perp}$, the orthogonal complement of N. For $\varepsilon = 0$, $u = v_0 \in N$. Let $\phi \in N \cap C_2$. $\int_0^{2\pi}\int_0^{\pi} \phi(u_{tt} - u_{xx})\, dx\, dt$ $= \int_0^{2\pi}\int_0^{\pi} u(\phi_{tt} - \phi_{xx})\, dx\, dt = 0 = \int_0^{2\pi}\int_0^{\pi} \phi(-\varepsilon F(x,t,u))\, dx\, dt$ where we integrated by parts and used our boundary and periodicity conditions. Since $C_2 \cap N$ is dense in N, $F(x,t,u) \perp N$, i.e. $\int_0^{2\pi}\int_0^{\pi} F(x,t,u)\phi\, dx\, dt = 0$ for all $\phi \in N$. Letting $\varepsilon \to 0$, we get

$$F(x,t,v_0) \perp N, \qquad v_0 \in N.$$

This is our bifurcation equation. We must use this equation to find v_0. Actually we will solve a more general bifurcation problem. Assume $w \in H_{k+1}$ is known. We will find $v \in H_k$ such that $F(x,t,v+w) \perp N$. We will need to

solve such a generalized bifurcation problem in §4. The solution of the bifurcation equation will be by a variational argument using a priori estimates and compactness. Before giving the proof we need some observations.

Remark 1: $\phi \in N$ implies $\phi = p(x+t) - p(-x+t) \equiv p^+ - p^-$. p is not uniquely determined since $(p^+ + c) - (p^- + c) = \phi$ for any constant c. We can normalize p by requiring $[p^+] = \int_0^{2\pi}\int_0^{\pi} p(x+t)\, dx\, dt = 0$ ($= [p^-]$ since $[\phi] = 0$ for it has no constant term in its Fourier expansion).

Remark 2: If $f(x+t), g(-x+t) \in H_0$, and if $[f] = 0$ or $[g] = 0$, then $(f,g) = 0$.

Proof: $f(x+t) = \sum_{-\infty}^{\infty} f_j\, e^{ij(x+t)}$, $g(-x+t) = \sum_{-\infty}^{\infty} g_k\, e^{ik(-x+t)}$.

$$(f,g) = \int_0^{2\pi}\int_0^{\pi} (\sum f_j e^{ij(x+t)})\ (\sum \bar{g}_k\, e^{-ik(-x+t)})\ dx\, dt$$

$$= \int_0^{\pi} (\sum_{j,k} \int_0^{2\pi} f_j\, \bar{g}_k\, e^{i(j+k)x}\, e^{i(j-k)t}\, dt)\ dx$$

$$= \int_0^{\pi} 2\pi \sum_j f_j\, \bar{g}_j\, e^{i\cdot 2jx}\, dx = 2\pi^2\, f_0\, \bar{g}_0$$

$$= 2\,[f]\,[\bar{g}] = 2[f]\,\overline{[g]}\ .$$

Thus $[f] = 0$ or $[g] = 0$ implies $(f,g) = 0$,

Remark 3: $\phi = p^+ - p^- \in N$ and $[p^+] = 0$ implies $p^+ \perp p^-$. This follows from Remark 2.

We can now begin our solution of the bifurcation equation.

Theorem 1: If $F(x,t,u) \in C^1$ in its arguments, $F_u \geq \beta > 0$, and w is given and smooth, there exists a unique $v \in N \cap H_1$ such that

$$\int_0^{2\pi}\int_0^{\pi} F(x,t,v+w)\phi \, dx \, dt = 0 \quad \text{for all } \phi \in N .$$

Moreover

$$\frac{\beta}{4} \sup_{x,t} |v(x,t)| \leq \sup_{x,t} |F(x,t,w(x,t))|$$

and

$$\beta|v|_1 \leq \text{const.} \ (|F_t(x,t,v+w)| + |F_u(x,t,v+w)w_t|) .$$

We will not prove Theorem 1 directly but will replace it by an approximate problem which we will solve. We will show the solution to the approximate problem v_K depending on the parameter K satisfies Theorem 1 if K is large enough.

Let $H(x,t,u)$ be such that $H_u = F$. Let $N_K = \{\phi \in N \mid |\phi|_1 \leq K\}$. N_K is a compact subset of H_0 for the elements of N_K are uniformly bounded and uniformly Holder continuous of order 1/2.

(To see this, note first that if $\phi \in N$, we can assume via Remark 3 that $|\phi|^2 = |p^+|^2 + |p^-|^2$.

$$|\phi(x_1,t_1)-\phi(x_2,t_2)| \le |p(x_1+t_1)-p(x_2+t_2)| + |p(-x_1+t_1)-p(-x_2+t_2)|$$

$$\le \left|\int_{x_2+t_2}^{x_1+t_1} p^{+'}(a)\, da\right| + \left|\int_{-x_2+t_2}^{-x_1+t_1} p^{-'}(a)\, da\right|$$

$$\le \text{const.}\, (|(x_1+t_1) - (x_2+t_2)|^{1/2} + |-x_1+t_1+x_2-t_2|^{1/2})|\phi|_1$$

and

$$|\phi(x,t)| = |\sum a_j \sin jx\, e^{ijt}|$$

$$\le \sum |a_j| \le (\sum \frac{1}{j^2})^{1/2}(\sum j^2|a_j^2|)^{1/2}$$

$$\le \text{const.}\ |\phi|_1^2.)$$

<u>Theorem 2</u>: <u>There exists</u> $v_K \in N_K$ <u>minimizing</u>

$$\int_0^{2\pi}\int_0^{\pi} H(x,t,v+w)\, dx\, dt \quad \text{for } \phi \in N_K\ .$$

<u>Moreover</u>

$$\frac{\beta}{4} \sup |v_K(x,t)| \le \sup_{x,t} |F(x,t,w(x,t))|$$

<u>and</u>

$$\rho|v_K|_1 \le \text{const.}(|F_t(x,t,v_K+w)| + |F_u(x,t,v_K+w)w_t|).$$

Proof. The proof consists of several steps:

(A) We show there exists v_K satisfying the minimum problem.

(B) We introduce a notion of admissible variations for the minimum problem and derive an "Euler equation."

(C) We derive a criterion for a variation to be admissible .

(D) We select two variations, show they are admissible, and

use them in our "Euler equation" to get the inequalities given in the statement of Theorem 2. This is really the heart of the proof.

Proof of A: We can assume $w = 0$ for if not we can replace F by $\tilde{F}(x,t,v) = F(x,t,v+w)$ in the argument that follows.

Let $\Phi(a) = \int_0^{2\pi}\int_0^{\pi} H(x,t,\phi)\, dx\, dt$. It is easily seen Φ is continuous on N_K in the H_0 topology. Since N_K is a compact set in H_0, Φ has a minimum on N_K which is assumed by some $v_K \in N_K$. We seek to show for K sufficiently large v_K is an interior minimum.

B: Let ϕ be chosen so that $v_K + \theta\phi \in N_K$ for every scalar $\theta < 0$ sufficiently close to 0. (The set of such ϕ's belong to a cone.) Such a ϕ will be termed an <u>admissible variation</u> for our minimization problem. Since v_K minimizes Φ,

$$\int_0^{2\pi}\int_0^{\pi} (H(x,t,v_K) - H(x,t,v_K+\theta\phi))\, dx\, dt \leq 0 .$$

Thus by the mean value theorem,

$$0 \geq \int_0^{2\pi}\int_0^{\pi} F(x,t,v_K+\theta\rho(x,t)\phi)(-\theta\phi)\, dx\, dt$$

where $|\rho(x,t)| \leq 1$. Dividing by $-\theta$ and letting $\theta \to 0$, $\theta\rho\phi \to 0$ uniformly and we obtain

$$\int_0^{2\pi}\int_0^{\pi} F(x,t,v_K)\phi\, dx\, dt \leq 0$$

for all admissible variations ϕ. This is the "Euler equation" for our problem.

C: ϕ is an admissible variation if $|v_K + \theta\phi|_1^2 \le K^2$,

$$|v_K + \theta\phi|_1^2 = \int_0^{2\pi}\int_0^{\pi} [(v_K + \theta\phi)^2 + (v_K + \theta\phi)_x^2 + (v_K + \theta\phi)_t^2]\, dx\, dt$$

$$= |v_K|_1^2 + 2\theta(v_K, \phi)_1 + \theta^2 |\phi|_1^2 \le K^2 .$$

A sufficient condition for this to occur is

$$2\theta(v_K, \phi)_1 + \theta^2 |\phi|_1^2 \le 0 .$$

Since $\theta < 0$, we can achieve this for all sufficiently small θ if $(v_K, \phi)_1 > 0$ for the linear term in θ dominates if θ is small enough.

D: To obtain the pointwise estimate of Theorem 2, we show a certain nonlinear function of v_K is admissible and then use it in the Euler equation.

Let

$$\begin{aligned} q(\lambda) &= 0 , & |\lambda| &\le M \\ &= \lambda - M, & \lambda &\ge M \\ &= -\lambda + M, & \lambda &\le -M \end{aligned}$$

$$v_K(x,t) \equiv v_K^+ - v_K^- .$$

By construction $q(v_K^+) - q(v_K^-) \in N$. We claim this is an admissible variation. All we need show is $(v_K, q(v_K^+) - q(v_K^-))_1 > 0$.

We take $M = 1/2 \sup_{x,t} |v_K^+(x,t)| = \frac{1}{2} \sup_{x,t} |v_K^-(x,t)|$.

If $M = 0$, $v_K \equiv 0$ and the estimates are trivially satisfied. Thus we can assume $M > 0$.

$$(v_K^+ - v_K^-, q(v_K^+)-q(v_K^-))_1 = \int_0^{2\pi}\int_0^{\pi} [(v_K^+-v_K^-)(q(v_K^+)-q(v_K^-))$$

$$+ (v_{Kx}^+ - v_{Kx}^-)(q'(v_K^+)v_{Kx}^+ - q'(v_K^-)v_{Kx}^-)$$

$$+ (v_{Kt}^+ - v_{Kt}^-)(q'(v_K^+)v_{Kt}^+ - q'(v_K^-)v_{Kt}^-)]\ dx\ dt\ .$$

By Remark 1, we can assume $[v_K^+] = [v_K^-] = 0$,

$$(v_K^+ - v_K^-)(q(v_K^+) - q(v_K^-)) = v_K^+q(v_K^+) + v_K^-q(v_K^-)$$

$$- (v_K^+q(v_K^-) + v_K^-q(v_K^+)).$$

Since $[v_K^+] = [v_K^-] = 0$, the last two terms vanish on integration. Since q is a monotonic odd function of its argument $v_K^{\pm}q(v_K^{\pm}) > 0$ at some point since $M > 0$. Then

$$\int_0^{2\pi}\int_0^{\pi} (v_K^+q(v_K^+) + v_K^-q(v_K^-))\ dx\ dt > 0\ .$$

The second term is

$$(v_{Kx}^+ - v_{Kx}^-)(q'(v_K^+)v_{Kx}^+ - q'(v_K^-)v_{Kx}^-) =$$

$$= q'(v_K^+)(v_{Kx}^+)^2 + q'(v_K^-)(v_{Kx}^-)^2 - v_{Kx}^+(q'(v_K^-)v_{Kx}^-)$$

$$- v_{Kx}^-(q'(v_K^+)v_{Kx}^+)\ .$$

The first two terms here are pointwise ≥ 0 since $q' \geq 0$. Since $[v_{Kx}^+] = [v_{Kx}^-] = 0$, the second two terms vanish on integration by Remark 2. A similar observation can be made

about the expression involving the t derivatives.

Thus $(v_K^+ - v_K^-, q(v_K^+) - q(v_K^-))_1 > 0$ and $(q(v_K^+)-q(v_K^-))$ is an admissible variation.

By our Euler equation:

$$\int_0^{2\pi}\int_0^{\pi} F(x,t,v_K+w)(q(v_K^+) - q(v_K^-))\, dx\, dt \leq 0 \ .$$

By the mean value theorem,

$$F(x,t,v_K) = F(x,t,0) + F_u(\text{intermediate point})v_K.$$

Thus

$$\int_0^{2\pi}\int_0^{\pi} F_u(\text{int.pt.})(v_K^+-v_K^-)(q(v_K^+)-q(v_K^-))$$

$$\leq -\int_0^{2\pi}\int_0^{\pi} F(x,t,0)(q(v_K^+) - q(v_K^-))\, dx\, dt$$

$$\leq \sup_{x,t} |F(x,t,0)| \int_0^{2\pi}\int_0^{\pi} (|q(v_K^+)| + |q(v_K)|)\, dx\, dt \ .$$

On the other hand, since q is monotonic, $(v_K^+-v_K^-)(q(v_K^+)-q(v_K^-)) \geq 0$. Thus using the monotonicity of F,

$$\int_0^{2\pi}\int_0^{\pi} F_u(\text{int. pt.})(v_K^+-v_K^-)(q(v_K^+)-q(v_K^-))\, dx\, dt \geq$$

$$\geq \beta \int_0^{2\pi}\int_0^{\pi} (v_K^+ - v_K^-)(q(v_K^+) - q(v_K^-))\, dx\, dt$$

$$= \beta \int_0^{2\pi}\int_0^{\pi} (v_K^+ q(v_K^+) + v_K^- q(v_K^-))\, dx\, dt$$

$$\geq \beta M \int_0^{2\pi}\int_0^{\pi} (|q(v_K^+)| + |q(v_K^-)|)\, dx\, dt \quad ,$$

since $\lambda q(\lambda) \geq M|q(\lambda)|$.

Dividing both sides of the inequality by the integral,

$$\beta M = \frac{\beta}{2} \sup_{x,t} |v_K^{\pm}| \leq \sup_{x,t} |F(x,t,\ 0)|$$

Since $|\sup|v_K|| \leq 2 \sup |v_K^+|$, we get the first inequality of Theorem 2 (with $w = 0$). Thus we have a pointwise estimate for v_K independent of K.

Let $\zeta^h(x,t) = \frac{1}{h}(\zeta(x,t+h)-\zeta(x,t))$. We claim $\phi = -(v_K^h)^{-h}$ is an admissible variation. It suffices to show $(v_K,-(v_K^h)^{-h})_1 > 0$. But $(v_K,-(v_K^h)^{-h})_1 = |v_K^h|_1^2 > 0$. Now by our Euler equation,

$$\int_0^{2\pi}\int_0^{\pi} F(x,t,v_K)(-v_K^h)^{-h}\, dx\, dt \leq 0$$

or

$$\int_0^{2\pi}\int_0^{\pi} (F(x,t,v_K))^h v_K^h\, dx\, dt \leq 0$$

$$(F(x,t,v_K))^h = F_t(\text{int.pt.}) + F_u(\text{int.pt.})(v_K)^h$$

by the mean value theorem. Our inequality becomes:

$$\int_0^{2\pi}\int_0^{\pi} F_u(\text{int.pt.})(v_K^h)^2\, dx\, dt \leq -\int_0^{2\pi}\int_0^{\pi} [F_t(\text{int.pt.})v_K^h]dx\, dt$$

Thus $\beta|v_K^h| \leq |F_t(\text{int.pt.})|$. Letting $h \to 0$, we get the inequality

$$\beta|v_{Kt}| \leq |F_t(x,t,v_K)|.$$

Recalling that for $\phi \in N$, $|\phi_t| = |\phi_x|$, we get the second inequality of Theorem 2 and Theorem 2 is proved.

Proof of Theorem 1: The two inequalities of Theorem 2 imply that for K sufficiently large, v_K is an interior point of N_K. Then $-\phi$ will be an admissible variation along with ϕ and our Euler equation becomes an equality:

$$\int_0^{2\pi}\int_0^{\pi} F(x,t,v_K+w)\phi \, dx \, dt = 0 \quad \text{for all admissible variations } \phi.$$

But now all $\phi \in N \cap H_1$ are admissible. Since $N \cap H_1$ is dense in N, the Euler equation is true for all $\phi \in N$.

v_K is unique in $N \cap H_1$ for if u_K is another solution,

$$0 = \int_0^{2\pi}\int_0^{\pi} (F(x,t,v_K+w) - F(x,t,u_K+w))(v_K-u_K) \, dx \, dt \geq \beta |v_K-u_K|^2$$

using the mean value theorem. Thus $v_K = u_K = v$ for k large enough and

$$\int_0^{2\pi}\int_0^{\pi} F(x,t,v+w)\phi \, dx \, dt = 0 \quad \text{for all } \phi \in N.$$

The estimates for v follow from Theorem 2.

We will conclude §2 by obtaining greater regularity for v. First we must get a pointwise estimate for v_t.

Since $F(x,t,v+w) \perp N \cap H_1$, $\frac{\partial}{\partial t} F = F_t + F_u(v_t+w_t) \perp N$

for if $\phi \in N$,

$$\int_0^{2\pi}\int_0^{\pi} (\tfrac{\partial}{\partial t} F)\phi \, dx \, dt = - \int_0^{2\pi}\int_0^{\pi} F \, \phi_t \, dx \, dt = 0 ,$$

$\frac{\partial}{\partial t}$ mapping $H_1 \cap N \to N$ with $H_1 \cap N$ dense in N. Thus

$$\int_0^{2\pi}\int_0^{\pi} (F_t + F_u(v_t + w_t))\phi = 0 \quad \text{for all } \phi \in N.$$

Take $\phi = q(v_t^+) - q(v_t^-) \in N$.

$$\int_0^{2\pi}\int_0^{\pi} F_u(v_t^+ - v_t^-)\ (q(v_t^+) - q(v_t^-))\ dx\ dt =$$

$$= - \int_0^{2\pi}\int_0^{\pi} (F_t + F_u w_t)(q(v_t^+) - q(v_t^-))\ dx\ dt \le$$

$$\le \sup_{s,t}\ [\,|F_t(x,t,v+w)| + |F_u(x,t,v+w)w_t|\,] \int_0^{2\pi}\int_0^{\pi}(|q(v_t^+)|+|q(v_t^-)|)dx\ dt$$

while

$$\int_0^{2\pi}\int_0^{\pi} F_u(v_t^+ - v_t^-)(q(v_t^+)-q(v_t^-))\ dx\ dt \ge$$

$$\ge \beta \int_0^{2\pi}\int_0^{\pi} (v_t^+ - v_t^-)(q(v_t^+)-q(v_t^-))\ dx\ dt \ge$$

$$\ge \beta M \int_0^{2\pi}\int_0^{\pi} (|q(v_t^+)| + |q(v_t^-)|)\ dx\ dt$$

as before. This leads to the estimate

$$\frac{\beta}{2} \sup_{x,t} |v_t^+| \le \sup_{x,t}[\,|F_t(x,t,v+w)|+|F_u(x,t,v+w)w_t|\,]\ .$$

Next take $\phi = (v_t^h)^{-h}$.

$$\int_0^{2\pi}\int_0^{\pi} (F_t + F_u(v_t + w_t))^h v_t^h\ dx\ dt = 0 = \int_0^{2\pi}\int_0^{\pi} [F_t^h v_t^h + F_u(x,t+h,v(x,t+h) +$$

$$+ w(x,t+h))[(v_t^h)^2 + w_t^h v_t^h] + F_u^h(v_t + w_t)v_t^h]\ dx\ dt.$$

$$\beta |v_t^h|^2 \le \beta \int_0^{2\pi}\int_0^{\pi} [F_t^h v_t^h + F_u(x,t+h,\ldots)w_t^h v_t^h + F_u^h(v_t+w_t)v_t^h]dx\ dt$$

$$\le |F_t^h + F_u(x,t+h,\ldots)w_t^h + F_u^h(v_t+w_t)|\,|v_t^h|$$

Letting $h \to 0$, we get $v_{tt} \in H_0$ and

$$\beta|v_{tt}| \le |F_{tt}| + 2|F_{tu}(v_t+w_t)| + |F_u w_{tt}| + |F_{uu}(v_t+w_t)^2|.$$

Continuing in this fashion, we can obtain estimates for higher t derivatives of v both in H_0 and pointwise. The higher

order x derivatives are obtained via $v_{tt} = v_{xx}$. We will not obtain pointwise estimates for the higher order derivatives since we will not use them in the later sections. However we will make the H_0 estimates precise.

<u>Theorem 3: For</u> $2 \le r \le k$, $|v|_r \le \hat{c}\,(|v|_{r-1} + |w|_r + 1)$ <u>where</u> $v_{t^{r-1}} = \frac{\partial^{r-1} v}{\partial t^{r-1}}$. $\hat{c} = \hat{c}(r, \|w\|_1, \|v_t\|_0)$ <u>is a constant depending in a monotone manner on</u> r, $\|w\|_1$, $\|v_t\|$.

The proof is given in the Appendix.

<u>Corollary</u>: For $2 \le r \le k$, $|v|_r \le \tilde{c}(|w|_r + 1)$ where $\tilde{c} = \tilde{c}(r, \|w\|_1, \|F(x,t,w)\|, \|F_t(x,t,v+w)\|, \|F_u(x,t,v+w)\|)$ is a constant depending in a monotone manner on its arguments.

Proof: By Theorem 3, we can estimate $|v|_r$ in terms of $|w|_r$, $|v|_{r-1}$, and $\hat{c}(r, \|w\|_1, \|v_t\|)$. $|v|_{r-1}$ can in turn be estimated by lower order derivative of v. Thus we can estimate $|v|_r$ by $|w|_r$, $\|w\|_1$, and $\|v_t\|$. We have shown earlier $\|v_t\|$ can be estimated by $\|F_t(x,t,v+w)\|$ and $\|F_u(x,t,v+w)w_t\|$ and $\|v\|$ by $\|F(x,t,w)\|$. Therefore we can get an estimate of the form stated above.

§3. Solution of a linear problem.

Before we can solve our nonlinear partial differential equation, we must solve a certain related linear equation. We need the following result.

<u>Theorem 4</u>: <u>Let</u> $f(x,t) \in \Gamma_k$ <u>and</u> $f \in N^{\perp}$. <u>Then there exists a unique</u> $w \in \Gamma_{k+1} \cap \overset{\circ}{H}_1 \cap N^{\perp}$ <u>such that</u> $w_{tt} - w_{xx} = f$ <u>and</u> $|w|_{k+1} \leq c_k |f|_k$.

Proof: We sketch the proof first. Using Fourier series we can find $w \in \overset{\circ}{H}_1 \cap N^{\perp}$ which is a weak solution to the equation. Using difference quotients in the t direction from the weak form of the equation we get more t differentiability for w. To get $w_{xx} \in H_0$ we cannot use global difference quotients with respect to x because of the boundary. However we can use local difference quotients to get w_{xx} in the interior. Then we can use the differential equation to get w_{xx} globally. The higher x derivatives can be obtained from the partial differential equation via a bootstrap operation.

Let $\Box = \dfrac{\partial^2}{\partial t^2} - \dfrac{\partial^2}{\partial x^2}$. We note that $\{\sin jx\, e^{int}\}$ is dense in H_0 and in $\overset{\circ}{H}_1$. $f = \sum_{j=1}^{\infty} \sum_{\substack{n=-\infty \\ j \neq |n|}}^{\infty} f_{jn} \sin jx\, e^{int}$.

Taking

$$w = \sum_{j=1}^{\infty} \sum_{\substack{n=-\infty \\ j \neq |n|}}^{\infty} w_{jn} \sin jx\, e^{int}, \text{ we find } w_{jn} = \frac{f_{jn}}{j^2 - n^2} .$$

$$|w|^2 = \pi^2 \sum_{j \neq |n|} \frac{|f_{jn}|^2}{(j^2 - n^2)^2} \leq \pi^2 \sum |f_{jn}|^2 = |f|^2$$

$$|w_x|^2 + |w_t|^2 = \pi^2 \sum \frac{j^2 + n^2}{(j^2 - n^2)^2} |f_{jn}|^2 \equiv \sum_{j > |n|} + \sum_{j < |n|} .$$

If $j > |n|$, $j^2 + n^2 < 2j^2$ while $(j^2 - n^2)^2 = (j-n)^2(j+n)^2 \geq j^2$. Thus $\sum_{j>|n|} \frac{j^2+n^2}{(j^2-n^2)^2} |f_{jn}|^2 \leq 2 \sum_{j>|n|} |f_{jn}|^2$. Similarly $j < |n|$ implies $(j^2 - n^2)^2 \geq n^2$. Consequently we get $|w|_1^2 \leq \text{const.}|f|^2$ and $w \in \mathring{H}_1 \cap N^{\perp}$. Also $(\Box\phi, w) = (\phi, f)$ for all $\phi \in C_0^\infty$.

If we replace f by f^h in the partial differential equation, the corresponding solution in $\mathring{H}_1 \cap N^{\perp}$ is w^h for the above inequality implies uniqueness of the solution and from $(\Box\phi^{-h}, w) = -(\Box\phi, w^h) = (\phi^{-h}, f) = -(\phi, f^h)$, we see w^h is a solution. Thus $|w^h|_1 \leq \text{const.}|f^h|$. Letting $h \to 0$, we get $w_t \in \mathring{H}_1 \cap N$ and $|w_t|_1 \leq \text{const.}|f_t|$. Similarly $w_{t^k} \in \mathring{H}_1 \cap N^{\perp}$ and $|w_{t^k}|_1 \leq \text{const.}|f_{t^k}|$.

Now we must obtain more x differentiability for w. We do this locally. Pick $\zeta(x) \in C^\infty$ with supp $\zeta(x) \subset (0,\pi)$ and supp $\zeta(x+h) \subset (0,\pi)$. We will now take ϕ^h to denote the difference quotient with respect to x.

Consider $(\Box\phi, (\zeta w)^h) = (-\Box\phi^{-h}, \zeta w) = -(\zeta\Box\phi^{-h}, w)$

$\Box\zeta\phi^{-h} = \zeta\Box\phi^{-h} - 2\zeta_x\phi_x^{-h} - \zeta_{xx}\phi^{-h}$. Using the above relation,

$$(\Box\phi, (\zeta w)^h) = -(\Box\zeta\phi^{-h}, w) - 2(\zeta_x\phi_x^{-h}, w) - (\zeta_{xx}\phi^{-h}, w)$$

$$= -(\zeta\phi^{-h}, f) - 2(\zeta_x\phi_x^{-h}, w) - (\zeta_{xx}\phi^{-h}, w) .$$

Also

$$(\Box\phi, (\zeta w)^h) = -(\phi_t, (\zeta w)_t^h) + (\phi_t, (\zeta w)_x^h).$$

Equating the two expressions for $(\Box\phi, (\zeta w)^h)$,

$$(\phi_x, (\zeta w)_x^h) = (\phi_t, (\zeta w)_t^h) - (\zeta\phi^{-h}, f) - 2(\zeta_x\phi_x^{-h}, w) - (\zeta_{xx}\phi^{-h}, w).$$

Now take $\phi = (\zeta w)^h$.

$$|(\zeta w)_x^h|^2 = |(\zeta w)_t^h|^2 + ((\zeta w)^h, (\zeta f)^h) + 2((\zeta w)_x^h, (\zeta_x w)^h)$$

$$+ ((\zeta w)^h, (\zeta_{xx} w)^h) .$$

This implies $|(\zeta w)_x^h|^2 \leq a|(\zeta w)_x^h| + b$ where a, b are constants > 0 and independent of h. Letting $h \to 0$, it follows $(\zeta w)_{xx} \in H_0$. Let $A \subset (0,\pi)$ be compact. We can find $\zeta \in C^\infty$ satisfying the above support condition on ζ and such that $\zeta \equiv 1$ on A. Thus $\zeta w = w$ on A and w_{xx} exists and is in H_0 on any compact $A \subset (0,\pi)$. Let us now return to our weak form of the equation. $(\Box\phi, w) = (\phi, f)$ $A = \operatorname{supp} \phi$ is compact. Thus by our above results, $(\Box\phi, w) = (\phi, \Box w) = (\phi, g)$ for all $\phi \in C_0^\infty$. Thus $\Box w = f$ a.e. $w_{xx} = w_{tt} - f$ and $|w_{tt}| + |f| < +\infty$. Hence it follows $w_{xx} \in H_0$ and $w \in H_2$.

Now working from the differential equation and using the known t differentiability of w, it follows via a bootstrap operation that $w \in H_{k+1}$ and $|w|_{k+1} \leq c_k |f|_k$.

§4. Solution of the nonlinear partial differential equation.

We want to solve $u_{tt} - u_{xx} + \varepsilon F(x,t,u) = 0$. We will find a solution of the form $u(x,t;\varepsilon) = v(x,t;\varepsilon) + \varepsilon w(x,t;\varepsilon)$ where $v \in N$ and $w \in N^{\perp}$. We employ an iteration scheme and show convergence for ε sufficiently small by a contraction argument.

Theorem 5: If $F(x,t,u) \in C_k$ in its arguments and is 2π periodic in t, for ε sufficiently small, the partial differential equation

$$u_{tt} - u_{xx} + \varepsilon F(x,t,u) = 0$$

has a solution $u \in H_k$ such that $u(x,t+2\pi) = u(x,t)$ and $u(0,t) = u(\pi,t) = 0$ provided that $F_u \geq \beta > 0$.

Proof: Let $u_0(x,t) = u(x,t;0)$. For $\varepsilon = 0$, the partial differential equation reduces to $\Box u_0 = 0$. Thus u_0 is determined from the bifurcation equation $F(x,t,u_0) \perp N$, $u_0 \in N$. By Theorems 1 and 3 we can solve this for $u_0 \in H_k$.

Let $w_1 \in N^{\perp}$ be the solution of $\Box w_1 = - F(x,t,u_0)$. By Theorem 4, we can find $w_1 \in H_{k+1}$. Now let $u_1 = v_1 + \varepsilon w_1$. We determine $v_1 \in N$ such that $F(x,t,v_1) \perp N$. Since $w_1 \in H_{k+1}$, we can find $v_1 \in H_k \cap N$, again by Theorems 1 and 3.

Assume $u_{n-1} = v_{n-1} + \varepsilon w_{n-1}$ has been found. We obtain $u_n = v_n + \varepsilon w_n$ with $v_n \in N$, $w_n \in N^{\perp}$ by solving $\Box w_n = -F(x,t,u_{n-1})$ for w_n and then get v_n by requiring $F(x,t,u_n) \perp N$. If $v_{n-1} \in H_k$ and $w_{n-1} \in H_{k+1}$, by Theorems 1, 3 and 4, we find $v_n \in H_k$ and $w_n \in H_{k+1}$.

If u_n converges sufficiently strongly, e.g. in C_2, to $u = v + \varepsilon w$, we will have

$$\Box w = - F(x, t, v + \varepsilon w)$$

or

$$\Box (v + \varepsilon w) + \varepsilon F(x, t, v + \varepsilon w) = 0$$

Thus it will satisfy our partial differential equation.

It remains to show the convergence of u_n. Assume for the moment that $\{|u_n|_k\}$ is bounded. Let

$$\delta w_n = w_{n+1} - w_n, \quad \delta v_n = v_{n+1} - v_n, \quad \delta u_n = \delta v_n + \varepsilon \delta w_n.$$

$$\Box \delta w_n = - (F(x,t,u_n) - F(x,t,u_{n-1})) .$$

Thus, by Theorem 4:

$$|\delta w_n|_1 \leq c_0 |F(x,t,u_n) - F(x,t,u_{n-1})|_0$$

$$= c_0 |F_u(x,t,\theta(x,t)u_n + (1-\theta)u_{n-1})\delta u_{n-1}|_0$$

where $\|\theta\| \leq 1$ (by the mean value theorem)

$$\leq c_0 \|F_u(x,t,\theta u_n + (1-\theta)u_{n-1}\| \; |\delta u_{n-1}|_0$$

$$= c_0 \|F_u(x,t,\theta u_n + (1-\theta)u_{n-1})\| (|\delta v_{n-1}|^2 + \varepsilon^2 |\delta w_{n-1}|^2)^{\frac{1}{2}}$$

since $\delta v_{n-1} \perp \delta w_{n-1}$.

Now we estimate $|\delta v_{n-1}|$ in terms of $|\delta w_{n-1}|$,

$$[F(x,t,u_n) - F(x,t,u_{n-1})] \perp N.$$

Thus

$$\int_0^{2\pi}\int_0^{\pi} [F(x,t,u_n) - F(x,t,u_{n-1})]\delta v_{n-1} \, dx \, dt = 0$$

$$= \int_0^{2\pi}\int_0^{\pi} F_u(\text{intermediate point})(\delta v_{n-1} + \varepsilon \delta w_{n-1})\delta v_{n-1} \, dx \, dt.$$

From this we get the estimate

$$\beta|\delta v_{n-1}| \leq \|F_u(\text{int. pt.})\| \ \varepsilon \ |\delta w_{n-1}|$$

and

$$|\delta w_n|_1 \leq c_0 \sup_{x,t;|u|\leq \sup_n \|u_n\|} |F_u(x,t,u)| \ \varepsilon \ |\delta w_{n-1}| \cdot$$

$$\cdot\left(1 + \frac{1}{\beta} \sup_{x,t;|u|\leq \sup\|u_n\|} |F_u(x,t,u)|^2\right)^{1/2}$$

(The boundedness of $\{|u_n|_k\}$ implies via Sobolev's inequality that $\{\|u_n\|\}$ is bounded.

Picking ε so small that $c_0\varepsilon \sup_{x,t;|u|\leq \sup\|u_n\|} |F_u(x,t,u)| \cdot (1+ \frac{1}{\beta} \sup_{x,t;|u|\leq \sup\|u_n\|} |F_u(x,t,u)|^2)^{1/2} \leq 1/2$, we get

$$|\delta w_n|_1 \leq \frac{1}{2} |\delta w_{n-1}|_0 \ .$$

Thus our mapping is contracting and $w_n \to w$, $v_n \to v$, $u_n \to u$ in $\overset{\circ}{H}_1$.

By the Nirenberg inequality [6], we get the interpolative estimate:

$$|\delta u_n|_j \leq C_j(|\delta u_n|_0^{1-\frac{j}{k}} |\delta u_n|_k^{j/k} + |\delta u_n|_0) \ .$$

It then follows w_n and v_n actually converge in H_j for all $j < k$ because of the boundedness of $\{|u_n|_k\}$. (By the Banach-Saks Theorem, it even follows $v \in H_k \cap \overset{\circ}{H}_1$ and $w \in H_{k+1} \cap \overset{\circ}{H}_1$).

To complete the proof, we must show the boundedness of $\{|u_n|_k\}$. We will make use of the composition of functions inequality due to J. Moser [7]:

$$|F(x,t,u(x,t))|_k \leq c(|u|_k + 1)$$

where $F \in C_k$ in its arguments, $u \in H_k$, and $c = c(k, \|u\|_0)$ where c depends monotonically on its arguments.
(Moser's result was proved for functions on a torus and was based on the Nirenberg and Hölder inequalities. By using the version of the Nirenberg inequality for functions in a bounded domain, Moser's proof easily generalizes to our case.)

We will also need the corollary of Theorem 3.

From Theorem 1, we have (since $w_0 = 0$)

$$\|v_0\| \leq \frac{4}{\beta} \|F(x,t,0)\| \leq F \frac{4}{\beta} \sup_{x,t,|u|\leq 1} |F(x,t,u)| \equiv B$$

and by the corollary to Theorem 3,

$$|v_0|_k \leq \tilde{c}\ (k, 0, \|F(x,t,0)\|, \|F_t(x,t,v_0)\|, \|F_u(x,t,v_0)\|)$$

$$\leq 2\tilde{c}\ (k, 1, B, \sup_{x,t;|u|\leq 2B} |F_t(x,t,u)|, \sup_{x,t;|u|\leq 2B} |F_u(x,t,u)$$

$$\equiv K ;$$

$$|w_1|_{k+1} \leq c_k |F(x,t,v_0)|_k \leq c_k\ c(k, \|v_0\|)(|v_0|_k + 1)$$

$$\leq c_k\ c\ (k, 2B)(2K+1)$$

$$\equiv M .$$

We need a separate estimate for $\|v_0\|$ (and $\|v_n\|$) since we impose no growth conditions on F. Assume

$$\|v_n\| \leq B, \quad |v_n|_k \leq K, \quad |w_n|_{k+1} \leq M .$$

$$|w_{n+1}|_{k+1} \leq c_k |F(x,t,u_n)|_k \leq c_k\ c(k, \|v_n\| + \varepsilon\|w_n\|)(|v_n|_k + \varepsilon|w_n|_k + 1)$$

$$\leq c_k\ c(k, B + \varepsilon\|w_n\|)(K + \varepsilon M + 1) .$$

By Sobolev's inequality $\|w_n\| \leq a|w_n|_2 \leq a\, M$.

Thus $|w_{n+1}|_{k+1} \leq c_k(k, 2B)(2K+1) = M$ for $\varepsilon M \leq K$, and $\varepsilon a M \leq B$.

From Theorem 1 we have

$$\|v_{n+1}\| \leq \frac{4}{\beta}\|F(x,t,\varepsilon w_n)\| \leq B \quad \text{provided} \quad \varepsilon a M \leq 1.$$

$$|v_{n+1}|_k \leq \tilde{c}(k, \varepsilon\|w_n\|_1, \|F(x,t,\varepsilon w_n)\|, \|F_t(x,t,v_{n+1}+\varepsilon w_{n+1})\|,$$

$$\|F_u(x,t,v_{n+1}+\varepsilon w_{n+1})\|(\varepsilon|w_{n+1}|_k + 1)$$

$$\leq 2\,\tilde{c}(k, 1, B, \sup_{x,t;|u|\leq 2B}|F_t(x,t,u)|, \sup_{x,t,|u|\leq 2B}|F_u(x,t,u)|$$

$$= K.$$

Thus our proof is complete.

§5. An expansion method.

In this section we given another method of proving the existence of solutions to our partial differential equation. We assume F is real analytic in u and C^{∞} in x and t. As before, F is 2π periodic in t and $F_u \geq \beta > 0$. We try for a solution of the form $u = \sum_{n=0}^{\infty} u_n \varepsilon^n$ where $u_n = v_n + w_n$, $v_n \in N$, $w_n \in N^{\perp}$. We will show in this section how to construct the u_n. In §6 the convergence of the series will be shown.

The main difficulty is the construction of u_0 which involves the solution of a bifurcation equation. We could use Theorem 1, however we choose to give an independent proof here. The method used is analogous to the proof of Theorem 1. There we used a calculus of variations argument approximating the problem by a compact one (Theorem 2). Here we use a Galerkin argument approximating the problem by a finite dimensional one. Unfortunately here we are unable to get pointwise estimates for the approximate problem. To get around this we must make the additional assumption $F_t(x,t,u)$ is bounded for all x,t,u. This is true in particular if $F(x,t,u) = G(u) + f(x,t)$.

Our partial differential equation is $u_{tt} - u_{xx} + \varepsilon F(x,t,u) = 0$. As in §2, the bifurcation condition is $F(x,t,u) \perp N$. Letting $\varepsilon \to 0$, we get $F(x,t,u_0) \perp N$. Setting the coefficient of ε^0 equal to 0 in the partial differential equation, we get $\Box w_0 = 0$ or $w_0 = 0$. Thus $u_0 = v_0 \in N$ and $F(x,t,v_0) \perp N$.

We use a Galerkin argument to solve this equation. We recall that

$$N = \{ \sum_{-\infty}^{\infty} a_k \sin kx\, e^{ikt} \mid \sum a_k^2 < +\infty \} .$$

Let

$$N_n = \{ \sum_{-n}^{n} a_k \sin kx\, e^{ikt} \} .$$

<u>Lemma 1</u>: Under our above conditions on F, there exists a unique $\phi_n \in N_n$ such that $F(x,t,\phi_n) \perp N_n$, i.e.

$$\int_0^{2\pi}\int_0^{\pi} F(x,t,\phi_n)\phi \, dx\, dt = 0 \quad \text{for all } \phi \in N_n .$$

Proof: Let H be such that $H_u = F$. Consider the problem

$$\underset{\phi \in N_n}{\text{Minimize}} \int_0^{2\pi}\int_0^{\pi} H(x,t,\phi)\, dx\, dt .$$

$F_u \geq \beta > 0$ implies H is a convex function of u. Therefore since our space is finite dimensional, there exists a minimizing function ϕ_n. The Euler equation for this problem is

$$\int_0^{2\pi}\int_0^{\pi} F(x,t,\phi_n)\phi \, dx\, dt = 0 \quad \text{for all } \phi \in N_n .$$

The uniqueness of ϕ_n is easily shown.

<u>Remark</u>: We can give another "Minty-type" proof of the lemma [8]. Let P_n be the projector on N_n. We seek $\phi_n \in N_n$ such that $P_n F(x,t,\phi_n) = 0$. N_n is a closed linear subspace of H_0 and $P_n \circ F$ is a continuous function on N_n. Moreover it is strongly monotonic for if $\phi, \psi \in N_n$, $(P_n F(x,t,\phi) - P_n F(x,t,\psi), \phi-\psi) =$ $(F(x,t,\phi)-F(x,t,\psi), \phi-\psi) = \int_0^{2\pi}\int_0^{\pi} F_u(\text{int. pt.})(\phi-\psi)^2\, dx\, dt \geq$

$\geq \beta\,|\phi-\psi|^2$. Therefore by the finite dimensional version of Minty's Theorem, there exists a unique $\phi_n \in H_n$ such that $P_n F(x,t,\phi_n) = 0$.

We will next show that the solutions to the finite problem converge to a solution to the bifurcation equation. We need a priori estimates for this.

<u>Theorem 6</u>: <u>If</u> $F_t(x,t,u)$ <u>is bounded, under our above conditions on</u> F, <u>there exists a unique</u> $v_0 \in N \cap \mathring{H}_1$ <u>such that</u> $F(x,t,v_0) \perp N$.

Proof: The Euler equation for ϕ_n of Lemma 1 is

$$\int_0^{2\pi}\int_0^{\pi} F(x,t,\phi_n)\phi \; dx\, dt = 0 \quad \text{for all } \phi \in H_n .$$

Note that differentiation with respect to t maps H_n into itself. Taking $\phi = \phi_{ntt}$ and integrating by parts:

$$\begin{aligned}\beta|\phi_{nt}|^2 &\leq \int_0^{2\pi}\int_0^{\pi} F_u(x,t,\phi_n)\phi_{nt}^2 \; dx\, dt \\ &= -\int_0^{2\pi}\int_0^{\pi} F_t(x,t,\phi_n)\phi_{nt} \; dx\, dt \\ &\leq |F_t(x,t,\phi_n)|\;|\phi_{nt}| .\end{aligned}$$

Thus $\beta|\phi_{nt}| \leq$ const. (independent of n). This implies $\{\phi_n(x,t)\}$ is uniformly bounded and equicontinuous as in §2. By Argela's theorem, there exists a subsequence $\phi_{n_i} \to v_0$ uniformly. v_0 is continuous and $v_0 \in N \cap \mathring{H}_1$. Also $F(x,t,\phi_{n_i}) \to F(x,t,v_0)$ uniformly. This implies $F(x,t,v_0) \perp N$, for let $\phi \in N_m$.

$$\int_0^{2\pi}\int_0^{\pi} F(x,t,\phi_{n_i})\phi \, dx\, dt = 0 \quad \text{for all } n_i \geq m \; .$$

Thus letting $n_i \to \infty$, $\int_0^{2\pi}\int_0^{\pi} F(x,t,v_0)\phi \, dx\, dt = 0$ for all $\phi \in N_m$ and hence for all $\phi \in N$. The uniqueness of v_0 follows from the monotonicity of F.

Remark: The uniqueness of v_0 implies that the entire sequence ϕ_n converges uniformly to v_0.

Theorem 7: $v_0 \in C^\infty$.

Proof: The solutions ϕ_n to the finite problem are all in C^∞. We will show their derivatives converge to the derivatives of v_0. Taking $\phi = \phi_{ntttt}$ in our Euler equation,

$$\int_0^{2\pi}\int_0^{\pi} F(x,t,\phi_n)\phi_{ntttt} \, dx\, dt = 0$$

$$= \int_0^{2\pi}\int_0^{\pi} (F_u\phi_{ntt}^2 + F_{uu}\phi_{nt}^2\phi_{ntt} + 2F_{ut}\phi_{nt}\phi_{ntt} + F_{tt}\phi_{ntt}) \, dx\, dt \; .$$

Using the monotonicity of F:

$$\beta|\phi_{ntt}|^2 \leq |\phi_{nt}^2||\phi_{ntt}|\|F_{uu}(x,t,\phi_n)\| + 2|\phi_{nt}||\phi_{ntt}|\|F_{ut}(x,t,\phi_n)\|$$

$$+ 2\pi^2|\phi_{ntt}|\|F_{tt}(x,t,\phi_n)\|$$

$$\leq \frac{\beta}{4}|\phi_{ntt}|^2 + \frac{4}{\beta}\|F_{uu}(x,t,\phi_n)\|^2|\phi_{nt}^2|^2 + \frac{\beta}{4}|\phi_{ntt}|^2$$

$$+ \frac{8}{\beta}|\phi_{nt}|^2\|F_{ut}(x,t,\phi_n)\|^2 + \frac{\beta}{4}|\phi_{ntt}|^2 +$$

$$+ \frac{4}{\beta}(\text{const.}\, \|F_{ut}(x,t,\phi_n)\|^2$$

where we used the inequality $ab \leq ca^2 + \frac{1}{c} b^2$ for $a,b,c > 0$. Since the functions ϕ_n are bounded pointwise independently of n, we can write

$$\frac{\beta}{4} |\phi_{ntt}|^2 \leq c_1 |\phi_{nt}^2|^2 + c_2 |\phi_{nt}|^2 + c_3$$

where c_1, c_2, c_3 are positive constants independent of n.

The boundedness of $\{|\phi_{nt}|\}$ implies the functions ϕ_n are uniformly Hölder continuous of order 1/2, i.e.

$$\|\phi_n\|_{1/2} = \sup_{(x,t) \neq (x_1,t_1)} \frac{|\phi_n(x,t) - \phi_n(x_1,t_1)|}{((x-x_1)^2 + (t-t_1)^2)^{1/4}}$$

is bounded by a constant independent of n, say $\|\phi_n\|_{1/2} \leq K$. In the Appendix it is shown if $\phi(x,t) \in C_2$ and 2π periodic in t, $|\phi_t^2|^2 \leq a\delta^{1/2} \|\phi\|_{1/2} |\phi_{tt}|^2 + b(\delta) \|\phi\|_{1/2}$, where a is a constant and $b(\delta)$ a function of δ and δ can be made arbitrarily small.

Thus

$$|\phi_{nt}^2|^2 \leq a\delta^{1/2} K |\phi_{ntt}|^2 + b(\delta) K$$

and

$$\frac{\beta}{4} |\phi_{ntt}|^2 \leq c_1 a\delta^{1/2} K |\phi_{ntt}|^2 + c_1 bK + c_2 |\phi_{nt}|^2 + c_3 .$$

Taking e.g. $\delta^{1/2} = \beta / 8 c_1 aK$, we get

$$\frac{\beta}{8} |\phi_{ntt}|^2 \leq \text{const. (independent of n)}.$$

Since $\phi_{ntt} = \phi_{nxx}$, it follows $v_0 \in C_1 \cap H_2$.

To obtain more differentiability for v_0 we multiply $F(x,t,\phi_n)$ by higher t derivatives of ϕ_n and use our Euler

equation as above. The most difficult estimate was the one we just made. (To make this precise we can use Theorem 3 with $v = \phi_n$ and $w = 0$.)

<u>Theorem 8</u>: <u>Under our above conditions on F, there exists a unique formal solution</u> $u = \sum_0^\infty u_n(x,t)\varepsilon^n$ <u>to</u> $u_{tt} - u_{xx} + \varepsilon F(x,t,u) = 0$ <u>satisfying our periodicity and boundary conditions. Moreover</u> $u_n = v_n + w_n$, $v_n \in N \cap C^\infty$, $w_n \in N^\perp \cap C^\infty$.

Proof: The uniqueness of u follows since our procedure is unique. We have already shown how to construct $u_0 = v_0 \in C^\infty \cap N$ and $w_0 = 0$.

Expanding the terms of our partial differential equation as a power series in ε and equating the coefficient of ε to 0, we get

$$\Box u_1 = \Box w_1 = - F(x,t,u_0) \ .$$

By Theorem 4, since $F(x,t,u_0) \in N^\perp \cap C^\infty$, we can find a unique $w_1 \in N^\perp \cap C^\infty$ satisfying this equation. Next equating the coefficient of ε^2 to 0,

$$\Box u_2 = - F_u(x,t,u_0)u_1 = - F_u(x,t,u_0)(w_1 + v_1) \ .$$

We require that the right hand side be orthogonal to N. This gives us a bifurcation equation for v_1. Since $F_u \geq \beta > 0$, this can be solved for $v_1 \in N$ by either argument of Lemma 1 or even more simply by the Lax-Milgram lemma. The regularity of v_1 would follow as in Theorem 7.

Assuming that u_{n-1} is known, we determine w_n by inverting

$$\square w_n = - \text{known function in } C^\infty \cap N^\perp$$

and find v_n by requiring the orthogonality condition needed for the next step be satisfied, i.e.

$$(\text{known } C^\infty \text{ function}) + F_u(x,t,u_0)(v_n + w_n) \perp N .$$

Therefore we have a formal solution to our partial differential equation.

<u>Remark</u>: If we had only assumed F to be C_k in x and t, we could have obtained a formal solution with $u_n \in H_k$ by the same methods.

§6. Convergence of the formal series.

We will use a majorant method to show that the formal series we have constructed for $u_{tt} - u_{xx} + \varepsilon F(x,t,u) = 0$ converges for ε sufficiently small. The assumption F_t is not needed here. A function $C(\varepsilon) = A(\varepsilon) + B(\varepsilon)$ will be obtained majorizing $u(x,t;\varepsilon) = v(x,t,\varepsilon) + w(x,t;\varepsilon)$ where A majorizes v and B majorizes w. By C majorizes u or $C \succ u$, we mean if $u = \sum u_n \varepsilon^n = v + w = \sum (v_n + w_n)\varepsilon^n$, and $C = \sum C_n \varepsilon^n$, then $|v_n|_1 + |w_n|_2 \le C_n$. Since by Sobolev's inequality we have $\|u_n\|_0 \le a(|v_n|_1 + |w_n|_2)$, u is majorized in the usual sense by aC ($u \prec aC$).

We will obtain a majorant problem not directly from the partial differential equation, but from the a priori estimates we obtain from it for v and w.

$$\Box u = \Box w = - \varepsilon F(x,t,u) \ .$$

Letting $F(x,t;u(x,t;\varepsilon)) = \sum_0^\infty f_n \varepsilon^n$, and equating coefficients of like powers of ε, we get

$$\Box w_n = - f_{n-1} \ .$$

From Theorem 4, we have

$$\frac{1}{a}\|w_n\|_0 \le |w_n|_2 \le c_1|f_{n-1}|_1 \le c_1(|f_{n-1}| + |f_{(n-1)x}| + |f_{(n-1)t}|).$$

Let

$$\begin{aligned} F(x,t,u) &= \sum_{m=0}^\infty F_m(x,t)(u-u_0)^m = \sum_m F_m\Big(\sum_{j=1}^\infty u_j \varepsilon^j\Big)^m \\ &= \sum_m F_m\Big(\sum_{j_i \ge 1} u_{j_1}\cdots u_{j_m}\, \varepsilon^{j_1+\cdots+j_m}\Big) \\ &= \sum_{m=0}^\infty \Big(\sum_{\substack{j_1+\cdots+j_m=n \\ m\le n}} F_m u_{j_1}\cdots u_{j_m}\Big)\varepsilon^n \ . \end{aligned}$$

Thus

$$f_n = \sum_{\substack{j_1+\dots+j_m=n \\ m\leq n}} F_m u_{j_1} \dots u_{j_m} .$$

If $F(x,t,u) \prec \bar{F}(u) = \sum \bar{F}_m (u-u_0)^m$ (where we consider F and $\bar{F}$ as functions of $u-u_0$ and majorization refers to the expansion about $(u-u_0)$, i.e. $|F_m(x,t)| \leq \bar{F}_m$, $\bar{F}_m$ a constant, and if $u \preceq C$, $u \prec aC$, then

$$|f_{n-1}|_0 = \Big| \sum_{\substack{j_1+\dots+j_m=n-1 \\ m\leq n-1}} F_m u_{j_1} \dots u_{j_m} \Big|_0$$

$$\leq \Big| \sum_{\substack{j_1+\dots+j_m=n-1 \\ m\leq n-1}} \bar{F}_m \, aC_{j_1} \dots aC_{j_m} \Big|_0$$

$$= c_2 \sum_{j_1+\dots+j_m=n-1} \bar{F}_m \, aC_{j_1} \dots aC_{j_m}$$

$$= c_2 \cdot \text{coefficient of } \varepsilon^{n-1} \text{ in expansion of } \bar{F}(aC).$$

(If $n = 0$, $u_0 = 0$, so we can ignore this case.)

$$f_{(n-1)x} = \frac{\partial}{\partial x} \text{ coef. of } \varepsilon^n \text{ in expansion of } F$$

$$= \quad " \quad " \quad " \quad " \quad " \qquad F_x + F_u u_x$$

$$F_x(x,t,u) = \sum_m F_{mx}(x,t)(u-u_0)^m = \sum_m \Big(\sum_{j_1+\dots+j_m=n} F_{mx} u_{j_1} \dots u_{j_m} \Big) \varepsilon^n .$$

Thus as above, assuming $\bar{F} \succ F_x$ (where we again consider $\bar{F}$ and F_x expanded about $u-u_0$),

$$|\text{coef. of } \varepsilon^{n-1} \text{ of } F_x|_0 \leq c_2 \text{ coef. of } \varepsilon^{n-1} \text{ of } \bar{F}(aC) .$$

$$F_u(x,t,u)u_x = \left(\sum_{m=1}^{\infty} m\, F_m(u-u_0)^{m-1}\right)\left(\sum_{p=0}^{\infty} u_{px}\varepsilon^p\right)$$

$$= \sum_{m=0}^{\infty} \left(\sum_{p+j_1+\dots+j_{m-1}=n} m\, F_m\, u_{j_1}\dots u_{j_{m-1}} u_{px}\right)\varepsilon^n$$

$$|\text{coef. of } \varepsilon^{n-1} \text{ of } F_u u_x|_0 = \left|\sum_{p+j_1+\dots+j_{m-1}=n-1} m\, F_m u_{j_1}\dots u_{j_{m-1}} u_{px}\right|_0$$

$$\leq \sum_{p+j_1+\dots+j_{m-1}=n} m\|F_m u_{j_1}\dots u_{j_{m-1}}\|\,|u_{px}|_0$$

$$\leq \sum m\, \bar{F}_m\, aC_{j_1}\dots aC_{j_{m-1}} C_p$$

$$= \text{coef. of } \varepsilon^{n-1} \text{ of } F_u(aC)C\ .$$

Treating the $f_{(n-1)t}$ term in a similar fashion and collecting our results, we get:

$$w \lesssim c_1\varepsilon\,[3c_2\,\bar{F}(aC) + 2\bar{F}_u(aC)C]\ .$$

We use the bifurcation equation to get a majorizing estimate for v. This is a bit tricky. First from $F(x,t,u) \perp N$, we get an estimate for $|v_n|_0$ in terms of previously estimated quantities. Then from $F_t(x,t,u) + F_u(x,t,u)u_t \perp N$, we obtain an estimate for $|v_{nt}|_0$ in terms of the estimate for $|v_n|_0$ and previously estimated quantities. We need two steps here because we imposed no growth conditions on F. The analog of this in §4 was the need to first estimate $\|v_n\|_0$ and then $|v_n|_k$ in the convergence proof there.

The bifurcation equation is $F(x,t,u) \perp N$ (formally).

$$F(x,t,u) = [F(x,t,u)-F(x,t,u_0)-F_u(x,t,u_0)(v-v_0)]+F(x,t,u_0)+$$
$$+ F_u(x,t,u_0)(v-v_0).$$

(Recall that $v_0 = u_0$.) Thus if $\phi \in N$,

$$\int_0^{2\pi}\int_0^{\pi} F_u(x,t,u_0)(v-v_0)\phi \, dx\, dt =$$

$$= - \int_0^{2\pi}\int_0^{\pi} [F(x,t,u)-F(x,t,u_0)-F_u(x,t,u_0)(v-v_0)]\phi \, dx\, dt$$

using that $F(x,t,u_0) \perp N$.

Equating coefficients of like powers of ε, and taking $\phi = v_n$, we get (for $n > 0$)

$$\beta|v_n|_0 \leq |\text{coef. of } \varepsilon^n \text{ of } F(x,t,u)-F(x,t,u_0)-F_u(x,t,u_0)(v-v_0)|_0$$

$$= |\sum_{j_1+\dots+j_m=n} F_m u_{j_1}\dots u_{j_m} - F_1 v_n|_0$$

$$\leq |\sum F_m u_{j_1}\dots u_{j_m} - F_1 u_n|_0 + \|F_1\| |w_n|_0$$

Let $G(x,t,u-u_0) = F(x,t,u) - F(x,t,u_0) - F_u(x,t,u_0)(u-u_0)$. Thus the expansion of G as a function of $(u-u_0)$ begins with quadratic terms. Let $\bar{G}(u-u_0) \succ G$, i.e. majorize as a function of $u-u_0$. $\bar{G}(u-u_0) = \sum_{n=2}^{\infty} \bar{G}_n(u-u_0)^n$.

Thus

$$\beta|v_n|_0 \leq |\sum \bar{G}_m aC_{j_1}\dots aC_{j_m}|_0 + \bar{F}_1 E_n$$

or

$$\beta|v_n|_0 \leq c_2 \cdot \text{coef. of } \varepsilon^n \text{ of } \bar{G}(a(C-C_0)) + \bar{F}_1 \text{ coef. of } \varepsilon^n \text{ of } B.$$

Differentiating the bifurcation equation with respect to t we get $F_t(x,t,u) + F_n(x,t,u)u_t \perp N$ (formally).

$$F_t + F_u u_t = F_u(x,t,u_0)(v_t - v_{0t}) + F_u(x,t,u_0)v_{0t}$$
$$+ (F_u(x,t,u) - F_u(x,t,u_0))(v_t - v_{0t}) + F_t(x,t,u_0)$$
$$+ (F_t(x,t,u) - F_t(x,t,u_0)) + F_u(x,t,u)w_t \ .$$

Since $F_u(x,t,u_0)v_{0t} + F_t(x,t,u_0) \perp N$, we find for $\phi \in N$,

$$\int_0^{2\pi}\int_0^{\pi} F_u(x,t,u_0)(v_t - v_{0t})\phi \, dx \, dt =$$
$$= - \int_0^{2\pi}\int_0^{\pi} [(F_u(x,t,u) - F_u(x,t,u_0))(v_t - v_{0t}) + F_t(x,t,u) - F_t(x,t,u_0) +$$
$$+ F_u(x,t,u)w_t]\phi \, dx \, dt \ .$$

Equating coefficients of like powers of ε and taking $\phi = v_{nt}$, we get (for $n > 0$)

$$\beta|v_{nt}| \leq |\text{coef. of } \varepsilon^n \text{ of right hand side}|_0 .$$

We will majorize the terms on the right hand side separately.

$$(F_u(x,t,u) - F_u(x,t,u_0))(v_t - v_{0t}) = \left(\sum_{m=1}^{\infty} m \, F_m (u-u_0)^{m-1}\right)\left(\sum_{p=1}^{\infty} v_{pt}\varepsilon^p\right)$$
$$= \sum_{n=1}^{\infty} \left(\sum_{p+j_1+\dots+j_{m-1}=n} m \, F_m u_{j_1} \dots u_{j_{m-1}} v_{pt}\right)\varepsilon^n$$

$$\left|\sum_{p+j_1+\dots+j_{m-1}=n} m \, F_m \, u_{j_1} \dots u_{j_{m-1}} v_{pt}\right| \leq \sum m \|F_m\|_0 \, aC_{j_1} \dots aC_{j_{m-1}} A_p$$
$$\leq \sum m \|F_m\|_0 \, aC_{j_1} \dots aC_{j_{m-1}} (A_p + B_p) \leq \sum m \, \bar{G}_m \, aC_{j_1} \dots aC_{j_{m-1}} (A_p + B_p)$$
$$= \text{coef. of } \varepsilon^n \text{ of } \bar{G}_u(a(C - C_0))(C - C_0) \ .$$

$$F_t(x,t,u) - F_t(x,t,u_0) = \sum_{m=1}^{\infty} F_{mt} u_{j_1} \cdots u_{j_m} \varepsilon^{j_1+\cdots+j_m}$$

$$= \sum_{n=1}^{\infty} \Big[\sum_{\substack{j_1+\cdots+j_m=n \\ j_m \neq n}} F_{mt}\, u_{j_1} \cdots u_{j_m} + F_{1t} w_n + F_{1t} v_n \Big] \varepsilon^n .$$

$$\Big| \sum_{\substack{j_1+\cdots+j_m=n \\ j_m \neq n}} F_{mt}\, u_{j_1} \cdots u_{j_m} + F_{1t}(v_n + w_n) \Big|_0$$

$$\leq c_2 \sum \bar{G}_m\, aC_{j_1} \cdots aC_{j_m} + \bar{F}_1 B_n + \bar{F}_1 |v_n|_0 ,$$

where we assumed $\bar{F} \succ F_t$ and $\bar{G} \succ F_t(x,t,u) - F_t(x,t,u_0) - F_u(x,t,u_0) \cdot (u-u_0)$.

Using our above estimate for v_n, we get

$$|\text{coef. of } \varepsilon^n \text{ of } F_t(x,t,u) - F_t(x,t,u_0)|_0$$

$$\leq \text{coef. of } \varepsilon^n \text{ of } \Big\{ c_2\, \bar{G}(a(C-C_0)) + \bar{F}_1 B + \frac{\bar{F}_1}{\beta} c_2 \cdot \bar{G}(a(C-C_0)) + \frac{\bar{F}_1^2}{\beta} B \Big\} .$$

In a similar fashion, we find

$$|\text{coef. of } \varepsilon^n \text{ of } F_u(x,t,u) w_t|_0 \leq \text{coef. of } \varepsilon^n \text{ of } \bar{F}_u(aC)B .$$

Combining these results, we have

$$\beta(v - v_0) \prec \bar{G}_u(a(C-C_0))(C-C_0) + [\bar{F}_1(1 + \frac{\bar{F}_1}{\beta}) + \bar{F}_u(aC)]B$$
$$c_2(1 + \frac{\bar{F}_1}{\beta})\, \bar{G}(a(C-C_0)) .$$

Now we construct our majorant equations. We can assume

C_0 is known and $C_0 \geq \|u_0\|$ and $\bar{F}_1 \geq \|F_u(x,t,u_0)\|$. Let $D = a(C-C_0) = a(A-A_0+B_0)$ (with $A_0 = C_0$).

Suppose $H(D) \succ 3c_2\bar{F}(D+aC_0) + D\bar{F}_u(D+aC_0)(\frac{D}{a} + C_0) + 1$ and $Q(D) \succ \bar{G}_u(D)\frac{D}{\beta a} + \frac{c_2}{\beta}(1 + \frac{\bar{F}_1}{\beta})\bar{G}(D)$. Then we get

$$w \preccurlyeq \varepsilon H(D)$$

$$v - v_0 \preccurlyeq Q(D) + cH(D)B \, .$$

We take $B = \varepsilon H(D)$. Thus we can substitute for B from the first equation and add:

$$u - u_0 \leq Q(D) + \varepsilon H(D) + c\varepsilon H(D)^2 \, .$$

Let

$$M = 3c_2 \sup_{x,t,|u-u_0|<2} [|F(x,t,u)|, |F_t|, |F_x|] +$$

$$+ 2 \sup_{x,t,|u-u_0|<2} |F_u(x,t,u)\frac{u}{a}| + 1 \, .$$

Then we can take $H(D) = M/(1-1/2D)$.

Let

$$\bar{M} = \frac{c_2}{\beta}(1 + \frac{\bar{F}_1}{\beta}) \sup_{x,t,|u-u_0|<1} \frac{|F(x,t,u)-F(x,t,u_0)-F_u(x,t,u_0)(u-u_0)|}{|u-u_0|^2}$$

$$+ \frac{1}{\beta a} \sup_{x,t,|u-u_0|<1} \frac{|F_u(x,t,u) - F_u(x,t,u_0)|}{|u-u_0|} \, .$$

We can take $Q(D) = \bar{M}D^2/(1-D)$. Thus we get

$$u-u_0 \preccurlyeq \frac{\bar{M}D^2}{1-D} + \varepsilon \left(\frac{M}{1-\frac{1}{2}D} + \frac{cM^2}{(1-\frac{1}{2}D)^2} \right) .$$

Since $\frac{1}{1-\frac{1}{2}D} \prec \frac{1}{1-D}$ and $\frac{1}{(1-\frac{1}{2}D)^2} \prec \frac{1}{1-D}$,

$$u - u_0 \ll \frac{\bar{M}\, D^2 + \varepsilon(M + cM^2)}{1-D} .$$

We take as our majorant equation

$$D = \frac{\bar{M}\, D^2 + \varepsilon(M + cM^2)}{1-D} .$$

$$D = \frac{-1+\sqrt{1-4\varepsilon(\bar{M}-1)(cM^2+M}}{2(\bar{M}+1)}$$

is an analytic function of ε for $|\varepsilon| < 1/(4(\bar{M}+1)(cM^2+M))$ and $D = 0$ for $\varepsilon = 0$.

If we expand D in the majorant equation as a power series in ε, we can recursively calculate the coefficients from A_0 and $B_0 = 0$ due to the way in which we set up the equation. For the same reason, the coefficients of D will be larger than the coefficients of u. Since we can solve the equations by another method and get an analytic solution for ε small, it follows our formal series for C and hence for u converges.

Appendix

We will give here the proof of Theorem 3 and the inequality of §5. The first proof follows from the Nirenberg inequality [6] and is based on an analogous proof of the composition of functions inequality for periodic functions due to J. Moser [7].

I. <u>Proof of Theorem 3.</u>

Let us first note that we have not shown the existence of derivatives of v of order higher than second. But this follows readily by the use of difference quotients as was done with v_{tt}.

We assume then that $v \in H_r$ and v is the component in N of a solution of our "generalized" bifurcation equation. We will show

$$|v|_r \leq \hat{c}(r, \|w\|_1, \|v_t\|)(|v|_{r-1} + |w|_r + 1) .$$

All r^{th} order derivatives of v are equal in norm since $v_{tt} = v_{xx}$. Thus it suffices to show

$$|v_{t^r}| \leq \hat{c}(r, \|w\|_1, \|v_t\|)(|v|_{r-1} + |w|_r + 1) .$$

$F(x,t,v+w) \perp N$ and $v + w \in H_r \cap C$ implies $\frac{\partial^r}{\partial t^r} F(x,t,v+w) \perp N$.

$$\int_0^{2\pi}\int_0^{\pi} F_u(x,t,v+w)v_{t^r}^2 = -\int_0^{2\pi}\int_0^{\pi} [(\frac{\partial^r}{\partial t^r} F(x,t,v+w) - F_u(x,t,v+w)(v_{t^r}+w_{t^r}) + F_u(x,t,v+w)w_{t^r}]\, dx\, dt .$$

Since $F_u \geq \beta > 0$, by Schwarz' inequality,

$$\beta|v_{t^r}| \leq \left|\frac{\partial^r}{\partial t^r} F(x,t,v+w) - F_u(x,t,v+w)(v_{t^r}+w_{t^r})\right|$$

$$+ |F_u(x,t,v+w)w_{t^r}|$$

$$\leq \left|\frac{\partial^r}{\partial t^r} F(x,t,v+w) - F_u(x,t,v+w)(v_{t^r}+w_{t^r})\right| + c|w_{t^r}|$$

where c depends on $\|v+w\|$.

Thus it suffices to show

$$\left|\frac{\partial^r}{\partial t^r} F(x,t,u) - F_u(x,t,u)u_{t^r}\right| \leq \hat{c}(r,\|w\|_1,\|v_t\|)(|v|_{r-1}+|w|_{r-1}+1)$$

where we write $u = v+w$. This inequality is true in general, i.e. it is not just true for solutions v of the bifurcation equation.

We estimate the left hand side of the inequality by using a symbolic expression for the chain rule and the Hölder and Nirenberg inequalities. Let $D = \partial/\partial t$.

$$D^r F(x,t,u) - F_u u_{t^r} = \sum_{\rho+\sigma\leq r}{}' \frac{\partial^{\rho+\sigma}}{\partial t^\rho \partial u^\sigma} F(x,t,u)\left(\sum_\alpha c_{\sigma\rho\alpha}(Du)^{\alpha_1} \cdots (D^{r-1}u)^{\alpha_{r-1}}\right),$$

where the ' indicates the $F_u u_{t^r}$ term is not present. $\alpha = (\alpha_1,\ldots,\alpha_{r-1})$ is a multiindex, $\sum \alpha_r = \rho$, $\sum r\,\alpha_r + \sigma = r$, and the $c_{\sigma\rho\alpha}$ are non-negative constants.

We can take as a typical term to estimate:

$$F_{t^\sigma u^\rho} \prod_{i=1}^{r-1} (D^i u)^{\alpha_i}$$

Let $u_0 = F_{t^\sigma u^\rho}$ and $u_i = D^i u$, $1 \leq i \leq r-1$. Take $p_i = \frac{r-i}{(i-1)\,\alpha_i}$, $i \geq 1$. $\sum_{i=1}^{r-1} \frac{1}{p_i} = \frac{1}{r-i} \sum_{i=1}^{r-1} (i-1)\alpha_i = \frac{r-\sigma-\rho}{r-2}$. We need $\sum_{i=0}^{r-1} \frac{1}{p_i} = 1$ to use the Hölder inequality. We can achieve this

if $\sigma + \rho \geq 2$ by taking $p_0 = \dfrac{i-2}{\sigma+\rho-2}$. If $\rho = 0$, $\sigma = r$ and we just have $|F_{t^r}| \leq B$ assuming $\|u_0\| \leq B$. If $\rho = 1$, we get terms of the form

$$|F_{t^{\sigma}u}\, u_{t^{r-1}}| \leq B|u_{t^{r-1}}| \leq B(|v_{t^{r-1}}|+|w_{t^{r-1}}|) .$$

Thus we can assume $\rho \geq 2$ and hence $\sigma+\rho \geq 2$ and take $p_0 = \dfrac{r-2}{\sigma+\rho-2}$.

Thus by the Hölder inequality we get:

$$\int_0^{2\pi}\int_0^{\pi} \prod_{i=0}^{r-1} u_i^{2\alpha_i}\, dx\, dt \leq \prod_{i=0}^{r-1} \left(\int_0^{2\pi}\int_0^{\pi} u_i^{2\alpha_i p_i}\, dx\, dt\right)^{1/p_i} .$$

By the Nirenberg inequality

$$\left(\int_0^{2\pi}\int_0^{\pi} (D^i u)^{2\left(\frac{r-2}{i-1}\right)}\, dx\, dt\right)^{\frac{i-1}{2(r-2)}} \leq \text{const.}\left(\|u_t\|_0^{1-\frac{i-1}{r-2}}\, |u|_{r-1}^{\frac{i-1}{r-2}} + |u_t|_0\right).$$

Since $|u_t|_0 \leq 2\pi^2\|u_t\|_0$ and $|u_t|_0 \leq |u|_{r-1}$,

$$\left(\int_0^{2\pi}\int_0^{\pi} (D^i u)^{2\frac{r-2}{i-1}}\, dx\, dt\right)^{\frac{i-1}{2(r-2)}} \leq c\|u_t\|^{1-\frac{i-1}{r-2}}\, |u|_{r-1}^{\frac{i-1}{r-2}} .$$

Substituting into our above product:

$$\int_0^{2\pi}\int_0^{\pi} \prod_{i=0}^{r-1} u_i^{2\alpha_i}\, dx\, dt \leq$$

$$\leq B^2 c^{2\sum_{i=1}^{r-1}\alpha_i}\, \|u_t\|^{2\sum_{i=1}^{r-1}\left(1-\frac{i-1}{r-2}\right)\alpha_i}\, |u|_{r-1}^{2\sum_{i=1}^{r-1}\frac{(i-1)\alpha_i}{r-2}}$$

$$\leq B^2 c^{2r}\left(\|u_t\|^{2\left(\rho-\frac{r-\sigma-\rho}{r-2}\right)}\, |u|_{r-1}^{2\left(\frac{r-\sigma-\rho}{r-2}\right)}\right.$$

$$\leq B^2 c^{2r}\, (\|u_t\|^{2r} + 1)(|u|_{r-1}^2 + 1) .$$

Finally combining terms we get

$$|D^r F(x,t,u) - F_u(x,t,u)u_{t^r}| \leq \hat{c}\ (r,\|u_t\|)(|u|_{r-1} + 1)$$

from which our inequality follows.

Remark: By using a similar but more complicated argument we can get the slightly better final estimate

$$\beta|v_{t^r}| \leq \hat{c}(r,\|w\|_0,\|v_t\|)\ (|v|_{r-1} + |w|_r + 1)$$

i.e. $\hat{c}$ depends only on $\|w\|_0$ rather than on $\|w\|_1$.

II. Proof of the inequality of §5.

Let $v(x,t) \in C_2$ in its variables, $0 \leq x \leq \pi$ and 2π periodic in t. We will denote the Hölder norm of order 1/2 of v by $\|v\|_{1/2}$.

$$\|v\|_{1/2} = \sup_{(x,t)\neq(x_1,t_1)} \frac{|v(x,t)-v(x_1,t_1)|}{((x-x_1)^2+(t-t_1)^2)^{1/4}} .$$

Let $\{\eta_i^2(t)\}$ be a finite partition of unity of $[0,2\pi]$ by continuous piecewise differentiable functions. Moreover let $\{\eta_i^2(t)\}$ be 2π periodic in t. Let the norm of the partition be $\leq \delta$. It is clear that for any δ, we can find such a partition, even a c^∞ partition.

We will obtain an inequality for v of the form

$$\int_0^{2\pi}\int_0^{\pi} v_t^4\ dx\ dt \leq c_1\delta^{1/2}\|v\|_{1/2}\int_0^{2\pi}\int_0^{\pi} v_{tt}^2\ dx\ dt + c_2(\delta)\|v\|_{1/2}$$

where $c_2(\delta)$ is inversely proportional to δ.

Consider $\int_0^{2\pi} \eta_i^2(t)\ v_t^4\ dt$. Integrating by parts and using the periodicity of η_i and v_t,

$$\int_0^{2\pi} \eta_i^2\ v_t^4\ dt = -3\int_0^{2\pi} \eta_i^2(t)\ [v(x,t)-v(x,\tau_i)]v_t^2\ v_{tt}\ dt$$
$$-2\int_0^{2\pi} \eta_i\ \eta_{it}[v(x,t)-v(x,\tau_i)]v_t^3\ dt$$

where $\tau_i \in \operatorname{supp} \eta_i^2$.

Thus

$$\int_0^{2\pi} \eta_i^2\ v_t^4\ dt \leq 3\|v\|_{1/2}\ \delta^{1/2}\int_0^{2\pi}(|\eta_i^2 v_t^2 v_{tt}|+|\eta_i\eta_{it}v_t^3|)\ dt$$
$$\leq 3\|v\|_{1/2}\ \delta^{1/2}\int_0^{-\pi}(\eta_i^2 v_t^4 + \eta_i^2 v_{tt}^2)\ dt$$
$$+ 3\|v\|_{1/2}\ \delta^{1/2}\int_0^{2\pi}\left[\frac{(\eta_i\eta_{it})^4}{4\varepsilon^4} + \varepsilon^{4/3}\frac{v_t^4}{4/3}\right] dt\ ,$$

where we used the Hölder inequality and $\varepsilon > 0$ is at our disposal.

Summing over i,

$$\int_0^{2\pi} v_t^4\ dt \leq 3\|v\|_{1/2}\ \delta^{1/2}\int_0^{2\pi}(v_t^4 + v_{tt}^2)\ dt$$
$$+ 3\|v\|_{1/2}\ \delta^{1/2}\ \varepsilon^{-4}\sum_i\int_0^{2\pi}(\eta_i\eta_{it})^4\ dt$$
$$+ 3\|v\|_{1/2}\delta^{1/2}\ \varepsilon^{4/3}\int_0^{-\pi} v_t^4\ dt\sum_i 1\ .$$

We can assume $\sum_i 1 \leq 2\delta^{-1}$. Taking $\varepsilon = \varepsilon_0\delta^{3/4}$,

$$\int_0^{2\pi} v_t^4 \, dt \le 3\|v\|_{1/2} \, \delta^{1/2} \int_0^{2\pi} (v_t^4 + v_{tt}^2) \, dt$$

$$+ 3\|v\|_{1/2} \, \delta^{-5/2} \varepsilon_0^{-4} \sum_i \int_0^{2\pi} (\eta_i \eta_{it})^4 \, dt$$

$$+ 6\|v\|_{1/2} \delta^{1/2} \, \varepsilon_0^{4/3} \int_0^{2\pi} v_t^4 \, dt \, .$$

$$(1 - 3\|v\|_{1/2} \, \delta^{1/2} - 6\|v\|_{1/2} \, \delta^{1/2} \, \varepsilon_0^{4/3}) \int_0^{2\pi} v_t^4 \, dt$$

$$\le 3\|v\|_{1/2} \, \delta^{1/2} \int_0^{2\pi} v_{tt}^2 \, dt + 3c_0 \|v\|_{1/2} \varepsilon_0^{-4} \, ,$$

where $c_0(\delta) = \sum_i \int_0^{2\pi} (\eta_i \eta_{it})^4 \, dt \; \delta^{-5/2}$.

Taking $\varepsilon_0^{4/3} = 1/2$ or $\varepsilon_0 = (1/2)^{3/4}$,

$$(1 - 6\|v\|_{1/2} \delta^{1/2}) \int_0^{2\pi} v_t^4 \, dt \le 3\|v\|_{1/2} \delta^{1/2} \int_0^{2\pi} v_{tt}^2 \, dt$$

$$+ 3c_0 \|v\|_{1/2} \, .$$

Taking δ so small that $1 - 6\|v\|_{1/2} \delta^{1/2} \ge 1/2$,

$$\int_0^{2\pi} v_t^4 \, dt \le 6\|v\|_{1/2} \, \delta^{1/2} \int_0^{2\pi} v_{tt}^2 \, dt + 6c_0(\delta) \|v\|_{1/2} \, .$$

Integrating the inequality over x, $0 \le x \le \pi$, we finally get

$$\int_0^{2\pi} \int_0^{\pi} v_t^4 \, dx \, dt \le c_1 \delta^{1/2} \|v\|_{1/2} \int_0^{2\pi} v_{tt}^2 \, dt + c_2(\delta) \|v\|_{1/2} \, .$$

Bibliography

[1] Stoker, J. J. Oscillations périodiques des systèmes non linéaires ayant une infinité de degres de liberté, Actes du Colloque International des Vibrations Non-Linéaires, Pub. Sci. Tech. Ministère de l'Air. No. 281, 1953.

[2] Ficken, F. A. and Fleishman, B. A. Initial value problems and time periodic solutions for a nonlinear wave equation, Comm. Pure Appl. Math. 10, 1957, pp. 331-356.

[3] Vejvoda, O. Periodic solutions of a linear and weakly nonlinear wave equation in one dimension, I, Czechoslavakian Mathematical Journal, 14, 1964, pp. 341-382.

[4] Prodi, G. Soluzioni periodiche de equazioni a derivate parziali di tipo iperbolico non lineari, Annali Mat. Pura Appl. 42, 1956, pp. 25-49.

[5] Cesari, L. Periodic solutions of hyperbolic partial differential equations, International Symposium on nonlinear differential equations and nonlinear mechanics, Academic Press, 1963, pp. 33-57.

[6] Nirenberg, L. On elliptic partial differential equations, Ann. Scuola Norm. Super. Pisa, Ser. 3, 13, 1959, pp. 1-48.

[7] Moser, J. Some aspects of nonlinear differential equations, 6 lectures, International Mathematical Summer Centeo, held at Varenna, Italy, 1964, to be published.

[8] Minty, G. Monotone (nonlinear) operators in Hilbert space, Duke Math. J. 29, 1962, pp. 341-346.

[9] Stewart, F. M. Periodic solutions of a nonlinear wave equation (Abstract), Bull. Amer. Math. Soc. 60, 1954, p. 344

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This paper concerns an existence proof of solutions of the partial differential equation

$$u_{tt} - u_{xx} + \varepsilon F(x,t,u) = 0 ,$$

where F is 2π periodic in t, under the boundary and periodicity conditions $u(0,t) = u(\pi,t) = 0$, $u(x,t+2\pi) = u(x,t)$. The main assumption is $\partial F/\partial u \geq \beta > 0$ and ε is sufficiently small. The main difficulty is solving an associated infinite dimensional bifurcation equation.

It is also shown how to find a solution by an expansion method if F is real analytic in u and a convergence proof for the formal series constructed is supplied.

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